An effective version of Nadkarni's thm

Joint w/ Alekos

(I)

Def$^{\underline{n}}$ – A countable Borel equiv. relation (CBER) is an equiv. rel. E on a standard Borel space X s.t. $E \subseteq X^2$ is Borel & all E-classes are ctbl.

– A Borel prob. measure μ on X is E-inv. if $\forall$ partial Borel bij. $f: A \to B$ s.t. $x E f(x)$, $\mu(A) = \mu(B)$. μ is E-ergodic if $\mu(A) \in \{0,1\}$ $\forall E$-inv. A.

Eg – E_0, $(\frac{1}{2}, \frac{1}{2})^{\mathbb{N}}$ unique (erg.) Borel inv. prob. meas.

– E_Γ, $\Gamma \curvearrowright 2^\Gamma$, $(\alpha, 1-\alpha)^\Gamma$ inv. erg. Borel p.m.

– E_t: no inv. prob. meas

pf Let μ be E_t-inv. Then $\mu(A) = 0 \Rightarrow \mu([A]_{E_t}) = 0$ (by F-M). ↯

Also, by $f(x) = 0^\frown x$, $\mu(2^{\mathbb{N}}) = \mu(0^\frown 2^{\mathbb{N}}) = 1$. So $0 = \mu(1^\frown 2^{\mathbb{N}}) = \mu([1^\frown 2^{\mathbb{N}}]_{E_t}) = \mu(2^{\mathbb{N}})$.

Def$^{\underline{n}}$ E is compressible if $\exists$ Borel compression, ie $f: X \to X$ s.t. f is inj, $x E f(x)$ $\forall x$, & $[X \setminus f(x)]_E = X$.

Obs E compressible $\Rightarrow$ E has no inv. Borel prob. meas.

Thm (Nadkarni '90, Becker–Kechris) E not compressible $\Rightarrow$ E has inv. Borel prob meas

Thm (Erg. decomp. thm Farrell, Varadarajan, '60s) E CBER on X, $INV_E \neq \emptyset$.

Then $EINV_E \neq \emptyset$ & $\exists \pi: X \to EINV_E$ Borel surj. s.t.

(i) π is E-inv

(ii) $e(\pi^{-1}(e)) = 1$ and e is the unique E-inv. erg. Borel prob meas. on $\pi^{-1}(e)$ for $e \in EINV_E$

(iii) $\forall \mu \in INV_E$, $\mu = \int \pi \, d\mu$

(II) <u>Effective descriptive set theory</u> ☆ Use coloured chalk

$X = \mathcal{N} = \mathbb{N}^{\mathbb{N}}$, $N_s = \{x \in \mathcal{N} : s \subseteq x\}$

arbitrary unions of N_s comp.   Suslin   "unif" continuum union of $\boldsymbol{\Pi}^0_1$ sets

$\boldsymbol{\Sigma}^0_1 \quad \boldsymbol{\Sigma}^0_2 \quad \dots \quad \text{Borel} = \boldsymbol{\Delta}^1_1 \quad \boldsymbol{\Sigma}^1_1 \quad \dots$

Hyp

$\boldsymbol{\Pi}^0_1 \quad \boldsymbol{\Pi}^0_2$ $\boldsymbol{\Pi}^1_1$

branches through arb trees   comp.   arb. ctbl union/intersection   "unif"

<u>Rmk</u> Only ctbly many $\Sigma^0_1, \Pi^0_1, \Delta^1_1, \dots$ sets

⇝ Makes sense to study complexity of pts. Eg $\alpha \in \Delta^1_1$ if $\{\alpha\} \in \Delta^1_1$.

⇝ Refinements of classical thms

<u>Thm</u> (Eff. perf. set thm, Harrison '67) Let $A \in \Sigma^1_1$. Then either $A \subseteq \{\alpha \in \mathcal{N} : \alpha \in \Delta^1_1\}$ or $\exists f : 2^{\mathbb{N}} \to A$ cont. embedding

<u>Rmk</u> Refinement by relativization

⇝ Extra tools to work w/

<u>Eg</u> "$\exists \alpha \in \Delta^1_1$" is Π^1_1 $\Rightarrow$ Simple pf that "$\exists !$" is Π^1_1, Lusin-Novikov

Used in original pf of E_0, G_0 dichotomies (⇝ only ctbly many Σ^1_1 sets)

<u>Thm</u> (Eff. General Glimm-Effros dich, HKL 90) E Δ^1_1 eq. rel. on $\mathcal{N}$. Either $\exists f : \mathcal{N} \to \mathcal{N}$ Δ^1_1 s.t. $x E y \Leftrightarrow f(x) = f(y)$ or $\exists g : 2^{\mathbb{N}} \to \mathcal{N}$ Borel inj. s.t. $x E_0 y \Leftrightarrow g(x) E g(y)$.

<u>Def^{n}</u> f is Δ^1_1 if graph(f) is Δ^1_1

<u>Rmk</u> Only half is effective

⇝ Bound complexity of things

<u>~~Thm~~ Eg</u> (Segal) E Δ^1_1, μ-hyp ⇝ $\exists$ $\Delta^1_1(\mu)$ witness

$\Rightarrow$ Bound on complexity of "E is μ-hyp"

$\Rightarrow$ [FKSU, 23+] Bound on complexity of subshift of X^{Γ} being meas. hyp.

(III) **Q** Is there an effective witness to compressibility?

A Yes! If E is Δ^1_1 and comp, $\exists$ Δ^1_1 compression

Thm (Eff. Nadkarni's thm, Kechris-W) E a Δ^1_1 CBER on $\mathcal{N}$. Then either

(i) E admits a Δ^1_1 compression or (ii) E admits an inv. Borel prob. meas.

Rmk Only half is effective

Cor (Eff. erg. decomp. thm, Kechris-W) E a Δ^1_1 CBER on $\mathcal{N}$, $INV_E \neq \emptyset$.

Then $EINV_E \neq \emptyset$ & $\exists$ Δ^1_1 E-inv. $Z \subseteq \mathcal{N}$ and $\pi : Z \to Prob(\mathbb{N}^{\mathbb{N}})$ s.t.

(i) $E|(\mathcal{N} \setminus Z)$ admits a Δ^1_1 compression

(ii) $\pi : Z \xrightarrow{\text{onto}} EINV_E$ is E-inv

(iii) $e(\pi^{-1}(e)) = 1$ & e is the unique ~~erg~~ E-inv. erg. Borel prob. meas. on $\pi^{-1}(e)$ for $e \in EINV_E$

(iv) $\forall \mu \in INV_E$, $\mu = \int \pi \, d\mu$

Here we identify $Prob(\mathbb{N}^{\mathbb{N}})$ w/ the set of all $\varphi \in [0,1]^{\mathbb{N}^{<\mathbb{N}}}$ s.t. $\varphi(\emptyset) = 1$ and $\varphi(s) = \sum_n \varphi(s^\frown n)$ (by $\mu \mapsto (s \mapsto \mu(N_s))$), which is Π^0_2.

(IV) **Q** When is other half effective?

Prop^n If E is a non-comp. CBER on $2^{\mathbb{N}}$ induced by a Δ^1_1 action of ctbly many Δ^1_1 homeomorphisms (compact act. realization), then $\exists\, \Delta^1_1$ E-inv. prob. meas.

Def^n Say a CBER F on Y is <u>inv. univ.</u> if $\forall$ CBERs E on sbs X, $E \sqsubseteq^i_B F$.

Rmk inv. univ. $\Rightarrow$ not compressible

Prop^n $\exists$ inv. univ. Δ^1_1 CBER on $\mathcal{N}$ w/ no Δ^1_1 inv. prob. meas.

"pf" Recall: $\{\alpha \in \mathcal{N} : \alpha \in \Delta^1_1\}$ is Π^1_1 $\rightsquigarrow$ $\exists$ ill-founded comp. tree T w/ no Δ^1_1 branches. Let F_0 be Δ^1_1 & inv. univ, F_1 comp. & Δ^1_1. Glue F_0, F_1 along T $\rightsquigarrow$ copy of F_0 at branches. No Δ^1_1 branches $\overset{F_1 \text{ comp.}}{\Rightarrow}$ no Δ^1_1 inv. meas. $\square$

Q [FKSV, 23+] Does every non-smooth aperiodic CBER admit a compact action realization?

A [FKSV, 23+] Not in "uniform/effective way"

<u>Concrete example</u> of Δ^1_1 CBER Borel iso' to compact subshift but w/ no Δ^1_1 iso.

E shift on $(2^\omega)^{\mathbb{F}_\infty}$, $Fr(\mathcal{C}^{\mathbb{F}_\infty})$ the free part.

pf of [FKSV, 23+] + fact from MOS. $\rightarrow$ Can find Δ^1_1, compact, E-inv. $K \subseteq \mathcal{C}^{\mathbb{F}_\infty}$ w/ Δ^1_1 iso' $E|K \cong E|Fr(\mathcal{C}^{\mathbb{F}_\infty})$.
$\rightsquigarrow$ admits Δ^1_1 inv. Borel prob. meas.

$F = {=_{\mathcal{N}}} \times E$ on $\mathcal{N} \times \mathcal{C}^{\mathbb{F}_\infty}$, $X = [T] \times Fr(\mathcal{C}^{\mathbb{F}_\infty})$.

$E|Fr(\mathcal{C}^{\mathbb{F}_\infty})$, $F|X$ inv. univ. for CBERs induced by free Borel act. of $\mathbb{F}_\infty$

$\Rightarrow E|Fr(\mathcal{C}^{\mathbb{F}_\infty}) \underset{\text{Borel}}{\cong} F|X$, but $F|X$ has no Δ^1_1 inv. Borel prob. meas.

$\Rightarrow$ No Δ^1_1 iso of $F|X$ w/ Δ^1_1 compact subshift of $\mathcal{C}^{\mathbb{F}_\infty}$